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Different Aspects of Relativistic Toda Chain

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ABSTRACT

We demonstrate that the generalization of the relativistic Toda chain (RTC) is a special reduction of two-dimensional Toda Lattice hierarchy (2DTL). We also show that the RTC is gauge equivalent to the discrete AKNS hierarchy and the unitary matrix model. Relativistic Toda molecule hierarchy is also considered, along with the forced RTC. The simple approach to the discrete RTC hierarchy based on Darboux-Bäcklund transformation is proposed.

1 Introduction

Since the paper of Ruijsenaars [1], where has been proposed, the relativistic Toda chain (RTC) system was investigated in many papers [2]-[4]. This system can be defined by the equation:

$$
\begin{aligned}
\ddot{q}_n = & \ (1+\epsilon\dot{q}_n)(1+\epsilon\dot{q}_{n+1})\frac{\exp(q_{n+1}-q_n)}{1+\epsilon^2\exp(q_{n+1}-q_n)} - \\
- & \ (1+\epsilon\dot{q}_{n-1})(1+\epsilon\dot{q}_n)\frac{\exp(q_n-q_{n-1})}{1+\epsilon^2\exp(q_n-q_{n-1})}
\end{aligned}
\tag{1}
$$

which transforms to the ordinary (non-relativistic) Toda chain (TC) in the evident limit $\epsilon \to 0$. The RTC is integrable, which was discussed in different frameworks (see, for example, [2]-[4] and references therein). The RTC can be obtained as a limit of the general Ruijsenaars system [1].

In this paper we are going to review different Lax representations of the RTC, and to establish numerous realtions of it with many well-known integrable systems like AKNS, unitary matrix model etc. It is also shown that the RTC hierarchy can be embedded to the 2DTL hierarchy [5].

Besides, we discuss the forced RTC hierarchy and its finite analog, the relativistic Toda molecule. At the end of this short paper we describe the simple approach to discrete evolutions of the RTC which is based on the notion of the Darboux-Bäcklund transformations and can be considered as a natural generalization of the corresponding notion in the usual Toda chain theory.

This short paper is minded as a brief review. For more details we refer the reader to our lengthy paper [6].

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2 Lax representation for RTC

Let us describe the Lax representation for the standard RTC equation. The usual procedure to obtain integrable non-linear equations consists of the two essential steps:

i) To find appropriate spectral problem for the Baker-Akhiezer function(s).

ii) To define the proper evolution of this function with respect to isospectral deformations.

Lax representation by three-term recurrent relation. In the theory of the usual Toda chain the first step implies the discretized version of the Schrödinger equation (see [7], for example). In order to get the relativistic extension of the Toda equations, one should consider the following "unusual" spectral problem

$$\Phi_{n+1}(z) + a_n\Phi_n(z) = z\{\Phi_n(z) + b_n\Phi_{n-1}(z)\} \ , \qquad n \in \mathbb{Z} \tag{2}$$

representing a particular discrete Lax operator acting on the Baker-Akhiezer function $\Phi_n(z)$. This is a simple three-term recurrent relation (similar to those for the Toda and Volterra chains) but with "unusual" spectral dependence.

As for the second step, one should note that there exist *two* distinct integrable flows leading to the same equation (1). As we shall see below, the spectral problem (2) can be naturally incorporated into the theory of two-dimensional Toda lattice (2DTL) which describes the evolution with respect to two (infinite) sets of times $(t_1, t_2, ...)$, $(t_{-1}, t_{-2}, ...)$ (positive and negative times, in accordance with [5]). Here we describe the two particular flows (at the moment, we deal with them "by hands", i.e. introducing the corresponding Lax pairs by a guess) which lead to the RTC equations (1). The most simple evolution equation is that with respect to the first *negative* time and has the form

$$\frac{\partial \Phi_n}{\partial t_{-1}} = R_n\Phi_{n-1} \tag{3}$$

with some (yet unknown) R_n.

The compatibility condition determines R_n in terms of a_n and b_n $R_n = \frac{b_n}{a_n}$ and leads to the following equations of motion:

$$\frac{\partial a_n}{\partial t_{-1}} = \frac{b_n}{a_{n-1}} - \frac{b_{n+1}}{a_{n+1}}; \qquad \frac{\partial b_n}{\partial t_{-1}} = b_n\left(\frac{1}{a_{n-1}} - \frac{1}{a_n}\right) \tag{4}$$

In order to get (1), we should identify

$$a_n = \exp(-\epsilon p_n); \qquad b_n = -\epsilon^2 \exp(q_n - q_{n-1} - \epsilon p_n) \tag{5}$$

Note that in this parameterization the "Hamiltonian" R_n depends only on coordinates q_n's: $R_n = -\epsilon^2 \exp(q_n - q_{n-1})$.

Performing the proper rescaling of time in (4) we reach the RTC equation (1).

As we noted already, the evolution (3), which leads to the RTC equations is not the unique one. The other possible choice leading to the same equations is

$$\frac{\partial \Phi_n}{\partial t_1} = -b_n(\Phi_n - z\Phi_{n-1}) \tag{6}$$

The compatibility condition of (2) and (6) gives the equations

$$\frac{\partial a_n}{\partial t_1} = -a_n(b_{n+1} - b_n) \qquad \frac{\partial b_n}{\partial t_1} = -b_n(b_{n+1} - b_{n-1} + a_{n-1} - a_n) \tag{7}$$

This leads to the same RTC equation (1).

2×2 **matrix Lax representation.** The same RTC equation can be obtained from the matrix Lax operator depending on the spectral parameter [3] (generalizing the Lax operator for the TC [7]). Then the RTC arises as the compatibility condition for the following 2×2 matrix equations:

$$L_n^{(\mathrm{S})}\psi_n = \psi_{n+1} \quad , \quad \frac{\partial \psi_n}{\partial t} = A_n\psi_n \tag{8}$$

where

$$L_n^{(\mathrm{S})} = \begin{pmatrix} \zeta\exp(\epsilon p_n) - \zeta^{-1} & \epsilon\exp(q_n) \\ -\epsilon\exp(-q_n+\epsilon p_n) & 0 \end{pmatrix} \quad ; \quad \psi_n = \begin{pmatrix} \psi_n^{(1)} \\ \psi_n^{(2)} \end{pmatrix} \tag{9}$$

$$A_n = \begin{pmatrix} \epsilon^2\exp(q_n - q_{n-1}+\epsilon p_{n-1}) & -\epsilon\zeta^{-1}\exp(q_n) \\ \epsilon\zeta^{-1}\exp(-q_{n-1}+\epsilon p_{n-1}) & 1-\ \zeta^{-2}+\epsilon^2 \end{pmatrix} \tag{10}$$

One can easily reduce these equations to the system (2) and (3).

To conclude this section, we remark that L-operator (9), which determines the RTC is not unique; moreover, it is not the simplest one. Indeed, we shall see that there exists the whole family of the gauge equivalent operators, which contains more "natural" ones and includes, in particular, the well known operator generating the AKNS hierarchy. From general point of view, the whole RTC hierarchy is nothing but AKNS and vice versa.

3 RTC and unitary matrix model, AKNS, etc.

Now we are going to describe the generalized RTC hierarchy as well as its connection with some other integrable systems. We start our investigation from the framework of orthogonal polynomials

Unitary matrix model. It is well-known that the partition function τ_n of the unitary one-matrix model can be presented as a product of norms of the biorthogonal polynomial system [8]. Namely, let us introduce a scalar product of the form [1]

$$< A,B >= \oint \frac{d\mu(z)}{2\pi iz}\exp\Big\{\sum_{m>0}(t_m z^m - t_{-m}z^{-m})\Big\}A(z)B(\frac{1}{z}) \tag{11}$$

Let us define the system of polynomials biorthogonal with respect to this scalar product

$$< \Phi_n, \Phi_k^\star >= h_n\delta_{nk} \tag{12}$$

Then, the partition function τ_n of the unitary matrix model is equal to the product of h_n's:

$$\tau_n = \prod_{k=0}^{n-1} h_k \quad , \qquad \tau_0 \equiv 1 \tag{13}$$

The polynomials are normalized as follows:

$$\Phi_n(z) = z^n + \ldots + S_{n-1}, \qquad \Phi_n^\star(z) = z^n + \ldots + S_{n-1}^\star, \qquad S_{-1} = S_{-1}^\star \equiv 1 \tag{14}$$

[1] The signs of positive and negative times are defined in this way to get the exact correspondence with the times introduced in [5].

These polynomials satisfy the following recurrent relations:

$$\begin{aligned} \Phi_{n+1}(z) &= z\Phi_n(z) + S_n z^n \Phi_n^\star(z^{-1}) \\ \Phi_{n+1}^\star(z^{-1}) &= z^{-1}\Phi_n^\star(z^{-1}) + S_n^\star z^{-n}\Phi_n(z) \end{aligned} \tag{15}$$

and

$$\frac{h_{n+1}}{h_n} = 1 - S_n S_n^\star \tag{16}$$

The above relations can be written in several equivalent forms. First, it can be presented in the form analogous to (2):

$$\Phi_{n+1} - \frac{S_n}{S_{n-1}}\Phi_n = z\left\{\Phi_n - \frac{S_n}{S_{n-1}}(1 - S_{n-1}S_{n-1}^\star)\Phi_{n-1}\right\} \tag{17}$$

$$\Phi_{n+1}^\star - \frac{S_n^\star}{S_{n-1}^\star}\Phi_n^\star = z^{-1}\left\{\Phi_n^\star - \frac{S_n^\star}{S_{n-1}^\star}(1 - S_{n-1}S_{n-1}^\star)\Phi_{n-1}^\star\right\} \tag{18}$$

From the first relation and using (2) and (5), one can immediately read off

$$\frac{S_n}{S_{n-1}} = -\exp(-\epsilon p_n); \qquad \frac{h_n}{h_{n-1}} = -\epsilon^2 \exp(q_n - q_{n-1}) \tag{19}$$

Thus, the orthogonality conditions (12) lead exactly to the spectral problem for the RTC. We should stress that equations (17), (18) can be *derived* from the unitary matrix model.

Using the orthogonal conditions, it is also possible to obtain the equations which describe the time dependence of Φ_n, $\Phi_n^\star$. Differentiating (12) with respect to times t_1 , t_{-1} gives the evolution equations:

$$\frac{\partial \Phi_n}{\partial t_1} = \frac{S_n}{S_{n-1}}\frac{h_n}{h_{n-1}}(\Phi_n - z\Phi_{n-1}); \qquad \frac{\partial \Phi_n}{\partial t_{-1}} = \frac{h_n}{h_{n-1}}\Phi_{n-1} \tag{20}$$

$$\frac{\partial \Phi_n^\star}{\partial t_1} = -\frac{h_n}{h_{n-1}}\Phi_{n-1}^\star; \qquad \frac{\partial \Phi_n^\star}{\partial t_{-1}} = -\frac{S_n^\star}{S_{n-1}^\star}\frac{h_n}{h_{n-1}}(\Phi_n^\star - z^{-1}\Phi_{n-1}^\star) \tag{21}$$

(see general evolution equations with respect to higher flows below). The compatibility conditions give the following nonlinear evolution equations:

$$\frac{\partial S_n}{\partial t_1} = S_{n+1}\frac{h_{n+1}}{h_n}; \qquad \frac{\partial S_n}{\partial t_{-1}} = S_{n-1}\frac{h_{n+1}}{h_n} \tag{22}$$

$$\frac{\partial S_n^\star}{\partial t_1} = -S_{n-1}^\star\frac{h_{n+1}}{h_n}; \qquad \frac{\partial S_n^\star}{\partial t_{-1}} = -S_{n+1}^\star\frac{h_{n+1}}{h_n} \tag{23}$$

As a consequence, in the polynomial case,

$$\frac{\partial h_n}{\partial t_1} = -S_n S_{n-1}^\star h_n; \qquad \frac{\partial h_n}{\partial t_{-1}} = S_{n-1}S_n^\star h_n \tag{24}$$

These are exactly relativistic Toda equations written in somewhat different form. Indeed, from (24), (22), (23) and (16) one gets[2]

$$\frac{\partial^2}{\partial t_1^2}\log h_n = -\left(\frac{\partial}{\partial t_1}\log h_n\right)\left(\frac{\partial}{\partial t_1}\log h_{n+1}\right)\frac{\dfrac{h_{n+1}}{h_n}}{1 - \dfrac{h_{n+1}}{h_n}} + (n \to n-1) \tag{25}$$

[2]The same equation holds for t_{-1}-flow.

On the other hand, the RTC is a particular case of the 2DTL hierarchy. Indeed, let us introduce the key objects in the theory of integrable systems - the τ-functions, which are defined through the relation $h_n = \tau_{n+1}/\tau_n$. Then, with the help of (22)-(24), one can show that the functions τ_n satisfy the first equation of the 2DTL:

$$\partial_{t_1}\partial_{t_{-1}} \log \tau_n = -\frac{\tau_{n+1}\tau_{n-1}}{\tau_n^2} \tag{26}$$

Therefore, it is natural to assume that the higher flows generate the whole set of non-linear equations of the 2DTL in spirit of [5]. This is indeed the case.

This completes the derivation of the RTC from the unitary matrix model.

RTC versus AKNS and "novel" hierarchies. Now let us demonstrate the correspondence between RTC and AKNS system. We have already seen that the orthogonality conditions naturally lead to the 2×2 formulation of the problem generated by the unitary matrix model:

$$L^{(\mathrm{U})}\begin{pmatrix} \Phi_n \\ \Phi_n^\star \end{pmatrix} = \begin{pmatrix} \Phi_{n+1} \\ \Phi_{n+1}^\star \end{pmatrix} \quad , \quad L^{(\mathrm{U})} = \begin{pmatrix} z & z^n S_n \\ z^{-n} S_n^\star & z^{-1} \end{pmatrix} \tag{27}$$

$$\frac{\partial}{dt_1}\begin{pmatrix} \Phi_n \\ \Phi_n^\star \end{pmatrix} = \begin{pmatrix} -S_n S_{n-1}^\star & z^n S_n \\ z^{1-n} S_{n-1}^\star & -z \end{pmatrix}\begin{pmatrix} \Phi_n \\ \Phi_n^\star \end{pmatrix} \tag{28}$$

$$\frac{\partial}{dt_{-1}}\begin{pmatrix} \Phi_n \\ \Phi_n^\star \end{pmatrix} = \begin{pmatrix} z^{-1} & -z^{n-1} S_{n-1} \\ -z^{-n} S_n^\star & S_{n-1} S_n^\star \end{pmatrix}\begin{pmatrix} \Phi_n \\ \Phi_n^\star \end{pmatrix} \tag{29}$$

(Equations (28), (29) follow from (20)-(21) and the original spectral problem (15)). Put $\Phi_n \equiv z^{n/2-1/4} F_n$, $\Phi_n^\star \equiv z^{-n/2+1/4} F_n^\star$. Then the spectral problem (27) can be rewritten in the matrix form

$$L_n^{(\mathrm{AKNS})}\mathcal{F}_n = \mathcal{F}_{n+1} \quad , \quad \mathcal{F}_n \equiv \begin{pmatrix} F_n \\ F_n^\star \end{pmatrix} \tag{30}$$

where

$$L_n^{(\mathrm{AKNS})} = \begin{pmatrix} \zeta & S_n \\ S_n^\star & \zeta^{-1} \end{pmatrix} \quad , \quad \zeta \equiv z^{1/2} \tag{31}$$

This is the Lax operator for the discrete AKNS [9]. Obviously the evolution equations (28), (29) can be written in terms of F_n, $F_n^\star$ as

$$\frac{\partial \mathcal{F}_n}{\partial t_1} = A_n^{(1)}\mathcal{F}_n \quad , \qquad A_n^{(1)} = \begin{pmatrix} -S_n S_{n-1}^\star & \zeta S_n \\ \zeta S_{n-1}^\star & -\zeta^2 \end{pmatrix} \tag{32}$$

$$\frac{\partial \mathcal{F}_n}{\partial t_{-1}} = -\,A_n^{(-1)}\mathcal{F}_n \quad , \qquad A_n^{(-1)} = \begin{pmatrix} -\zeta^{-2} & \zeta^{-1} S_{n-1} \\ \zeta^{-1} S_n^\star & -S_{n-1} S_n^\star \end{pmatrix} \tag{33}$$

Note that after introducing the trivial flow

$$\frac{\partial \mathcal{F}_n}{\partial t_0} = A_n^{(0)}\mathcal{F}_n \quad , \qquad A_n^{(0)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{34}$$

we get the difference non-linear Schrödinger system (DNLS) [9] (see also [7]) generated by the "mixed" flow

$$\frac{\partial \mathcal{F}_n}{\partial T} \equiv \left(\frac{\partial}{\partial t_0} - \frac{\partial}{\partial t_{-1}} - \frac{\partial}{\partial t_1}\right)\mathcal{F}_n = (A_n^{(0)} + A_n^{(-1)} - A_n^{(1)})\mathcal{F}_n \equiv$$
$$\equiv \begin{pmatrix} 1 + S_n S^\star_{n-1} - \zeta^{-2} & \zeta^{-1} S_{n-1} - \zeta S_n \\ \zeta^{-1} S^\star_n - \zeta S^\star_{n-1} & -1 - S_{n-1} S^\star_n + \zeta^2 \end{pmatrix} \tag{35}$$

Indeed, from the compatibility conditions for (31), (35) or, equivalently, just from (22) (along with the trivial evolution $\partial_{t_0} S_n = 2S_n$, $\partial_{t_0} S^\star_n = -2S^\star_n$) one gets the discrete version of the nonlinear Schrödinger equation:

$$\frac{\partial S_n}{\partial T} = -(S_{n+1} - 2S_n + S_{n-1}) + S_n S^\star_n \big(S_{n+1} + S_{n-1}\big) \tag{36}$$

Note also that the "novel" hierarchy of [10] is equivalent to the RTC (and, therefore, to the AKNS hierarchy) as well. Namely, the Lax operator in [10], i.e.

$$\widehat{L}_n = \begin{pmatrix} z + u_n v_n & u_n \\ v_n & 1 \end{pmatrix} \quad ; \qquad \widehat{L}_n \begin{pmatrix} \phi_n^{(1)} \\ \phi_n^{(2)} \end{pmatrix} = \begin{pmatrix} \phi_{n+1}^{(1)} \\ \phi_{n+1}^{(2)} \end{pmatrix} \tag{37}$$

defines the recurrent relation of the form (17):

$$\phi_{n+1}^{(1)} - \left(u_n v_n + \frac{u_n}{u_{n-1}}\right)\phi_n^{(1)} = z\left(\phi_n^{(1)} - \frac{u_n}{u_{n-1}}\phi_{n-1}^{(1)}\right) \tag{38}$$

thus revealing the connection with the RTC. Comparing (17) and (38) leads to the identification $u_n = S_n h_n$, $v_n = \frac{S^\star_{n-1}}{h_n}$, where h_n's satisfy (16). Moreover, from (37) and (15) it is easy to see that $\phi_n^{(1)} = \Phi_n$; $\phi_n^{(2)} = \frac{1}{h_n}\Big(z^n \Phi^\star_n - S^\star_{n-1}\Phi_n\Big)$ and, therefore, $\widehat{L}_n$ can be obtained from $L_n^{(\mathrm{AKNS})}$ by the discrete gauge transformation:

$$\widehat{L}_n = U_{n+1} L_n^{(\mathrm{AKNS})} U_n^{-1}; \qquad U_n = z^{n/2-1/4}\begin{pmatrix} 1 & 0 \\ -\frac{S^\star_{n-1}}{h_n} & \frac{z^{1/2}}{h_n} \end{pmatrix}; \qquad z = \zeta^2 \tag{39}$$

Evolution equations (22)- (24) in terms of new variables $u_n,\ v_n$ have the form

$$\begin{aligned} \frac{\partial u_n}{\partial t_1} &= u_{n+1} - u_n^2 v_n \ , & \frac{\partial u_n}{\partial t_{-1}} &= \frac{u_{n-1}}{1 + u_{n-1} v_n} \\ \frac{\partial v_n}{\partial t_1} &= -v_{n-1} + u_n v_n^2 \ , & \frac{\partial v_n}{\partial t_{-1}} &= -\frac{v_{n+1}}{1 + u_n v_{n+1}} \end{aligned} \tag{40}$$

and easily reproduce the usual AKNS equations in the continuum limit since

$$\begin{aligned} \big(\partial_{t_0} - \partial_{t_1} - \partial_{t_{-1}}\big)u_n &= -\big(u_{n+1} - 2u_n + u_{n-1}\big) + \big(u_{n-1}^2 + u_n^2\big)v_n + \ldots \\ \big(\partial_{t_0} - \partial_{t_1} - \partial_{t_{-1}}\big)v_n &= \big(v_{n+1} - 2v_n + v_{n-1}\big) - \big(v_n^2 + v_{n+1}^2\big)u_n + \ldots \end{aligned} \tag{41}$$

We conclude with the remark that the operator $L_n^{(\mathrm{S})}$ in (9) is also gauge equivalent to $L_n^{(\mathrm{AKNS})}$ (see [6]).

Non-local Lax representation. There is another form of the recurrent relations which is non-local (i.e. contains all the functions with smaller indices) but instead expresses

$\Phi_n(z)$ through themselves. This form is crucial for dealing with the RTC as a particular reduction of the 2DTL. Let us introduce the normalized functions

$$\mathcal{P}_n(z) \equiv \Phi_n(z) \quad , \qquad \mathcal{P}^\star_n(z^{-1}) \equiv \frac{1}{h_n}\Phi^\star_n(z^{-1}) \tag{42}$$

From (17), (18) one can show that in the forced and fast-decreasing cases some proper solutions satisfy the equations

$$\begin{gathered} z\mathcal{P}_n(z) = \mathcal{P}_{n+1}(z) - S_n h_n \sum_{k=-\infty}^{n} \frac{S^\star_{k-1}}{h_k}\mathcal{P}_k(z) \equiv \mathcal{L}_{nk}\mathcal{P}_k(z) \\ z^{-1}\mathcal{P}^\star_n(z^{-1}) = \frac{h_{n+1}}{h_n}\mathcal{P}^\star_{n+1}(z^{-1}) - S^\star_n \sum_{k=-\infty}^{n} S_{k-1}\mathcal{P}^\star_k(z^{-1}) \equiv \overline{\mathcal{L}}_{kn}\mathcal{P}^\star_k(z^{-1}) \end{gathered} \tag{43}$$

This expression is correct for general (non-polynomial) $\mathcal{P}_n$ and $\mathcal{P}^\star_n$ provided the sums run over all integer k. In the polynomial case, the sums automatically run over only non-negative k. The last representation of the spectral problem will be useful to determine the general evolution of the system. Indeed, these relations manifestly describe the embedding of the RTC into the 2DTL [5,11], which is given essentially by *two* Lax operators ($\mathcal{L}$ and $\overline{\mathcal{L}}$).

RTC as reduction of 2DTL. In order to determine the whole set of the evolution equations, one can use different tricks. For example, one can use embedding (43) of the system into the 2DTL, making use of the standard evolution of this latter [5]. Let us briefly describe the formalism of the 2DTL following [5]. In their framework, one introduces *two* different Baker-Akhiezer (BA) $\mathbb{Z}\times\mathbb{Z}$ matrices $\mathcal{W}$ and $\overline{\mathcal{W}}$. These matrices satisfy the linear system:

i) the matrix version of the spectral problem:

$$\mathcal{L}\mathcal{W} = \mathcal{W}\Lambda \ , \quad \overline{\mathcal{L}}\,\overline{\mathcal{W}} = \overline{\mathcal{W}}\Lambda^{-1} \tag{44}$$

ii) the matrix version of the evolution equations:

$$\begin{gathered} \frac{\partial\mathcal{W}}{\partial t_m} = (\mathcal{L}^m)_+\mathcal{W} \ , \qquad \frac{\partial\overline{\mathcal{W}}}{\partial t_m} = (\mathcal{L}^m)_+\overline{\mathcal{W}} \\ \frac{\partial\mathcal{W}}{\partial t_{-m}} = (\overline{\mathcal{L}}^{\,m})_-\mathcal{W} \ , \qquad \frac{\partial\overline{\mathcal{W}}}{\partial t_{-m}} = (\overline{\mathcal{L}}^{\,m})_-\overline{\mathcal{W}} \ , \qquad m = 1,2,... \end{gathered} \tag{45}$$

where $\mathbb{Z}\times\mathbb{Z}$ matrices $\mathcal{L}$ and $\overline{\mathcal{L}}$ have (by definition) the following structure:

$$\begin{gathered} \mathcal{L} = \sum_{i\le 1} \mathrm{diag}[b_i(s)]\Lambda^i \ \ ; \quad b_1(s) = 1 \\ \overline{\mathcal{L}} = \sum_{i\ge -1} \mathrm{diag}[c_i(s)]\Lambda^i \ \ ; \quad c_{-1}(s) \neq 0 \end{gathered} \tag{46}$$

Here $\mathrm{diag}[b_i(s)]$ denotes an infinite diagonal matrix $\mathrm{diag}(\ldots\ b_i(-1),\ b_i(0),\ b_i(1),\ \ldots)$; Λ is the shift matrix with the elements $\Lambda_{nk} \equiv \delta_{n,k-1}$ and for arbitrary infinite matrix $A = \sum_{i\in\mathbb{Z}} \mathrm{diag}[a_i(s)]\Lambda^i$ we set

$$(A)_+ \equiv \sum_{i\ge 0} \mathrm{diag}[a_i(s)]\Lambda^i \ \ , \qquad (A)_- \equiv \sum_{i<0} \mathrm{diag}[a_i(s)]\Lambda^i \tag{47}$$

i.e. $(A)_+$ is the upper triangular part of the matrix A (including the main diagonal) while $(A)_-$ is strictly the lower triangular part.

Note that (46) can be written in components as

$$\mathcal{L}_{nk} = \delta_{n+1,k} + b_{k-n}(n)\theta(n-k)\ , \quad \overline{\mathcal{L}}_{nk} = c_{-1}(n)\delta_{n-1,k} + c_{k-n}(n)\theta(k-n)\ \ ; \quad n,k \in \mathbb{Z} \tag{48}$$

The compatibility conditions imposed on (44),(45) give rise to the infinite set (hierarchy) of nonlinear equations for the operators $\mathcal{L}$, $\overline{\mathcal{L}}$ or, equivalently, for the coefficients $b_m(n)$, $c_m(n)$. This is what is called 2DTL hierarchy.

From (43), one gets two matrices

$$\mathcal{L}_{nk} = \delta_{n+1,k} - \frac{h_n}{h_k} S_n S^\star_{k-1}\theta(n-k)\ , \qquad k,n \in \mathbb{Z} \tag{49}$$

$$\overline{\mathcal{L}}_{nk} = \frac{h_n}{h_{n-1}}\delta_{n-1,k} - S_{n-1}S^\star_k\theta(k-n)\ , \qquad k,n \in \mathbb{Z} \tag{50}$$

which have exactly the form (48). Now using the technique developed in [5], one can get the whole evolution of the RTC hierarchy. However, it can be also easily obtained in the framework of the orthogonal polynomials.

Evolution and orthogonal polynomials. Differentiating the orthogonality conditions with respect to arbitrary times, one can obtain with the help of (43) the evolution of polynomials $\mathcal{P}_n$ and $\mathcal{P}^\star_n$ [6]:

$$\begin{array}{ll} \dfrac{\partial \mathcal{P}_n}{\partial t_m} = -[(\mathcal{L}^m)_-]_{nk}\mathcal{P}_k; & \dfrac{\partial \mathcal{P}_n}{\partial t_{-m}} = [(\overline{\mathcal{L}}^{\,m})_-]_{nk}\mathcal{P}_k \\ \dfrac{\partial \mathcal{P}^\star_n}{\partial t_m} = -[(\mathcal{L}^m)_+]_{kn}\mathcal{P}^\star_k; & \dfrac{\partial \mathcal{P}^\star_n}{\partial t_{-m}} = [(\overline{\mathcal{L}}^{\,m})_+]_{kn}\mathcal{P}^\star_k \\ \dfrac{\partial h_n}{\partial t_m} = (\mathcal{L}^m)_{nn}h_n\ \ , & \dfrac{\partial h_n}{\partial t_{-m}} = -(\overline{\mathcal{L}}^{\,m})_{nn}h_n \end{array} \tag{51}$$

4 Forced RTC hierarchy

RTC-reduction of 2DTL. Let us formulate in some invariant terms what reduction of the 2DTL corresponds to the RTC hierarchy. Return again to the Lax representation (43) embedding the RTC into the 2DTL. Using (16), one can easily prove the following identities

$$\sum_{k=n}^{N} \frac{S_{k-1}S^\star_{k-1}}{h_k} = \frac{1}{h_N} - \frac{1}{h_{N-1}}; \qquad \sum_{k=n}^{N} S_k S^\star_k h_k = h_n - h_{N+1} \tag{52}$$

Because of these identities, the matrices $\mathcal{L}$ and $\bar{\mathcal{L}}^T$ have zero modes $\sim S_{k-1}$ and $S^\star_{k-1}/h_k$ respectively. Therefore, one could naively expect that they are not invertible and get (using (52)) that

$$(\mathcal{L}\bar{\mathcal{L}})_{nk} = \delta_{nk} - \frac{S_n S^\star_k h_n}{h_{-\infty}}; \qquad (\bar{\mathcal{L}}\mathcal{L})_{nk} = \delta_{nk} - \frac{S_{n-1}S^\star_{k-1}h_\infty}{h_k} \tag{53}$$

Since the reduction is to be described as an invariant condition imposed on $\mathcal{L}$ and $\bar{\mathcal{L}}$, these formulas might serve as a starting point to describe the reduction of the 2DTL to the RTC hierarchy only if their r.h.s. does not depend on the dynamical variables. It seems not to be the case.

However, these formulas require some careful treatment. Indeed, the formulation of the 2DTL in terms of infinite-dimensional matrices, although being correct as a formal construction requires some accuracy if one wants to work with the genuine matrices since the products of the infinite-dimensional matrices should be properly defined. In fact, this product exists for the "band" matrices, i.e. those with only a finite number of the non-zero diagonals, and in some other more complicated cases (of special divergency conditions). One can easily see from (43) that the RTC Lax operators do not belong to this class. Therefore, equations (53) just do not make sense in this case (this is why the interpretation of the general RTC hierarchy in invariant (say, Grassmannian) terms is a little bit complicated).

However, in the case of forced hierarchy, some of the indicated problems are removed since one needs to multiply only quarter-infinite matrices and, say, the product $\mathcal{L}\bar{\mathcal{L}}$ always exists. Certainly the inverse order of the multipliers is still impossible. Therefore, only the first formula in (53) becomes well-defined acquiring the form

$$(\mathcal{L}\bar{\mathcal{L}})_{nk} = \delta_{nk} \tag{54}$$

This formula can be already taken as a definition of the RTC-reduction of the 2DTL in the forced case as it does not depend on dynamical variables. Now we will show how this definition is reflected in different formulations of the 2DTL.

Determinant representation. One can show (see [6]) that τ-function of the forced hierarchy $\tau_n = 0,\ n < 0$ has the following determinant representation [12]

$$\tau_n(t) = \det \left[\partial_{t_1}^i (-\partial_{t_{-1}})^j \int_\gamma A(z,w) \exp\left\{ \sum_{m>0} (t_m z^m - t_{-m} w^{-m}) \right\} dz dw \right] \Bigg|_{i,j=0,\dots,n-1} \tag{55}$$

where the matrix A has the form $A(z,w) = \frac{\mu'(z)}{2\pi i z}\delta(z - w^{-1})$ [6].

Indeed, let us rewrite the orthogonality relation (12) in the matrix form. We define matrices D and $D^\star$ with the matrix elements determined as the coefficients of the polynomials $\Phi_n(z)$ and $\Phi_n^\star(z)$

$$\Phi_i(z) \equiv \sum_j D_{ij} z^{j-1}, \quad \Phi_i^\star(z) \equiv \sum_j D_{ij}^\star z^{j-1} \tag{56}$$

Then, (12) looks like

$$D \cdot C \cdot D^{\star T} = H \tag{57}$$

where superscript T means transponed matrix and H denotes the diagonal matrix with the entries $C_{ii} = h_{i-1}$ and C is the moment matrix with the matrix elements

$$C_{ij} \equiv \int_\gamma \frac{d\mu(z)}{2\pi i z} z^{i-j} \exp\left\{ \sum_{m>0} (t_m z^m - t_{-m} z^{-m}) \right\} \tag{58}$$

Let us note that D ($D^{\star T}$) is the upper (lower) triangle matrix with the units on the diagonal (because of (14)). This representation is nothing but the Riemann-Hilbert problem for the forced hierarchy. Now taking the determinant of the both sides of (57), one gets

$$\det_{n\times n} C_{ij} = \prod_{k=0}^{n-1} h_k = \tau_n \tag{59}$$

due to formula (13). The remaining last step is to observe that

$$C_{ij} = \partial_{t_1}^i (-\partial_{t_{-1}})^j \int_\gamma \frac{d\mu(z)}{2\pi i z} \exp\left\{ \sum_{m>0} (t_m z^m - t_{-m} z^{-m}) \right\} = \partial_{t_1}^i (-\partial_{t_{-1}})^j C_{11} \tag{60}$$

i.e.

$$\tau_n(t) = \det_{n\times n}\left[\partial_{t_1}^i(-\partial_{t_{-1}})^j \int_\gamma \frac{d\mu(z)}{2\pi iz}\exp\left\{\sum_{m>0}(t_m z^m - t_{-m}z^{-m})\right\}\right] \tag{61}$$

This expression coincides with (55). One can also remark that the moment matrix C_{ij} is Toeplitz matrix. This proves from the different approach that the RTC-reduction is defined by the Toeplitz matrices.

5 Relativistic Toda molecule

General properties. Let us consider further restrictions on the RTC which allows one to consider the *both* products in (53). Namely, in addition to the condition $\tau_n = 0,\ n < 0$ picking up forced hierarchy, we impose the following constraint

$$\tau_n = 0, \quad n > N \tag{62}$$

for some N. This system should be called $N-1$-particle relativistic Toda molecule, by analogy with the non-relativistic case and is nothing but RTC-reduction of the two-dimensional Toda molecule [13,14][3].

$sl(N)$ Toda can be described by the kernel $A(z, w)$

$$A(z,w) = \sum_k^N f^{(k)}(z)g^{(k)}(w) \tag{63}$$

where $f^{(k)}(z)$ and $g^{(k)}(z)$ are arbitrary functions. From this description, one can immediately read off the corresponding determinant representation (55).

Indeed, equation (26) and condition (62) implies that $\log\tau_0$ and $\log\tau_N$ satisfy the free wave equation $\partial_{t_1}\partial_{t_{-1}}\log\tau_0 = \partial_{t_1}\partial_{t_{-1}}\log\tau_N = 0$. Since the relative normalization of τ_n's is not fixed, we are free to choose $\tau_0 = 1$. Then, $\tau_0(t) = 1,\ \ \tau_N(t) = \chi(t_1)\bar\chi(t_{-1})$, where $\chi(t_1)$ and $\bar\chi(t_{-1})$ are arbitrary functions. 2DTL with boundary conditions (6) was considered in [13]. The solution to (26) in this case is given by [14]:

$$\tau_n(t) \ = \ \det\ \partial_{t_1}^{i-1}(-\partial_{t_{-1}})^{j-1}\tau_1(t) \tag{64}$$

with

$$\tau_1(t) = \sum_{k=1}^N a^{(k)}(t)\bar a^{(k)}(t_{-1}) \tag{65}$$

where functions $a^{(k)}(t)$ and $\bar a^{(k)}(t_{-1})$ satisfy

$$\det\partial_{t_1}^{i-1}a^{(k)}(t) = \chi(t), \qquad \det(-\partial_{t_{-1}})^{i-1}\bar a^{(k)}(t_{-1}) = \bar\chi(t_{-1})$$

This result coincides with that obtained by substituting into (55) the kernel $A(z, w)$ of the form (63).

Lax representation. In all our previous considerations, we dealt with infinite-dimensional matrices. Let us note that the Toda molecule can be effectively treated in terms of $N \times N$ matrices like the forced case could be described by the quarter-infinite matrices. This allows one to deal with the *both* identities (53) since all the products of *finite* matrices are well-defined.

[3]Sometimes the Toda molecule is called non-periodic Toda [15]. It is an immediate generalization of the Liouville system.

To see this, one can just look at the recurrent relation (43) and observe that there exists the finite-dimensional subsystem of (N) polynomials which is decoupled from the whole system. The recurrent relation for these polynomials can be considered as the finite-matrix Lax operator (which still does not depend on the spectral parameter, in contrast to (9)). Indeed, from (43) and condition (62), i.e. $h_n/h_{n-1} = 0$ as $n \geq N$ (the Toda molecule conditions in terms of S-variables read as $S_n = S^\star_n = 1$ for $n > N-2$ or $n > 0$), one can see that

$$z\mathcal{P}_N(z) = \mathcal{P}_{N+1}(z) - \mathcal{P}_N(z), \quad z\mathcal{P}_{N+1}(z) = \mathcal{P}_{N+2}(z) - \mathcal{P}_{N+1}(z) \quad \text{etc.} \tag{66}$$

i.e. all the polynomials $\mathcal{P}_n$ with $n > N$ are trivially expressed through $\mathcal{P}_N$. Therefore, the system can be effectively described by the dynamics of only some first polynomials (i.e. has really finite number of degrees of freedom). Certainly, all the same is correct for the star-polynomials $\mathcal{P}^\star_n$ although, in this case, it would be better to use the original non-singularly normalized polynomials $\Phi^\star_n$.

Now let us look at the corresponding Lax operators (49)-(50). They are getting quite trivial everywhere but in the left upper corner of the size $N \times N$ [6]. Therefore, one can restrict himself to the system of N polynomials $\mathcal{P}_n$, $n = 0, 1, \ldots, N-1$ and the finite matrix Lax operators (of the size $N \times N$).

Now one needs only to check that this finite system still has the same evolution equations (51). It turns out to be the case only for the first $N-1$ times. This is not so surprising, since, in the finite system with $N-1$ degrees of freedom, only first $N-1$ time flows are independent. Therefore, if looking at the finite matrix Lax operators, one gets the dependent higher flows. On the other hand, if one embeds this finite system into the infinite 2DTL, one observes that the higher flows can be no longer described inside this finite system. Just the finite system is often called relativistic Toda molecule.

To simplify further notations, we introduce, instead of S_n, $S^\star_n$, the new variables $s_n \equiv (-)^{n+1}S_n$, $s^\star_n \equiv (-)^{n+1}S^\star_n$. Then, one can realize that the Lax operator can be constructed as the product of simpler ones $\mathcal{L} = \mathcal{L}_N\mathcal{L}_{N-1}\ldots\mathcal{L}_1$, where $\mathcal{L}_k$ is the unit matrix wherever but a 2×2-block:

$$\mathcal{L}_k \equiv \begin{pmatrix} 1 & \vdots & \\ \cdots & G_k & \cdots \\ & \vdots & 1 \end{pmatrix} \qquad G_k \equiv \begin{pmatrix} s_k & 1 \\ s_k s^\star_k - 1 & s^\star_k \end{pmatrix} \tag{67}$$

Analogously, $\bar{\mathcal{L}} = \bar{\mathcal{L}}_1 \ldots \bar{\mathcal{L}}_{N-1}\bar{\mathcal{L}}_N$ with

$$\bar{\mathcal{L}}_k \equiv \begin{pmatrix} 1 & \vdots & \\ \cdots & \bar{G}_k & \cdots \\ & \vdots & 1 \end{pmatrix} \qquad \bar{G}_k \equiv \begin{pmatrix} s^\star_k & -1 \\ 1 - s_k s^\star_k & s_k \end{pmatrix} \tag{68}$$

One can trivially see that $\mathcal{L}_k\bar{\mathcal{L}}_k = \bar{\mathcal{L}}_k\mathcal{L}_k = 1$, and, therefore, one obtains $\mathcal{L}\bar{\mathcal{L}} = \bar{\mathcal{L}}\mathcal{L} = 1$ (cf. (53)).

From these formulas, one obtains that $\det\mathcal{L} = \det\bar{\mathcal{L}} = 1$ which reminds once more of the $sl(N)$ algebra. More generally, the factorization property of the Lax operators opens the wide road to the group theory interpretation of the RTC molecule – see [6].

6 Discrete evolutions and limit to Toda chain

Darboux-Bäcklund transformations. Now we are going to discuss some discrete evolutions of the RTC given by the Darboux-Bäcklund transformations and their limit to the usual Toda chain. One can easily take the continuum limit of the formulas of this section to reproduce the TC as the limit of the RTC, both with the standard continuous evolutions.

The discrete evolution equations in the RTC framework were recently introduced by [3] in a little bit sophisticated way. Here we outline the simple approach based on the notion of the Darboux-Bäcklund transformation (DBT). More details will be presented in the separate publication [16].

Let discrete index i denote the successive DBT's. The spectral problem now can be written as follows:

$$\Phi_{n+1}(i|z) + a_n(i)\Phi_n(i|z) = z\Big\{\Phi_n(i|z) + b_n(i)\Phi_{n-1}(i|z)\Big\} \tag{69}$$

Let us define the first forward DBT (treating it as a discrete analog of (3)):

$$\Phi_n(i+1|z) = \Phi_n(i|z) + \boldsymbol{\alpha}_n^{(1)}(i)\Phi_{n-1}(i|z) \tag{70}$$

where $\boldsymbol{\alpha}_n^{(1)}(i)$ are some unknown functions. One requires that $\Phi_n(i+1)$ satisfies the same spectral problem as (69) but with the shifted value of $i \to i-1$. Then the compatibility condition gives the equations of the discrete RTC:

$$a_n(i+1) = a_{n-1}(i)\,\frac{a_n(i) - \boldsymbol{\alpha}_{n+1}^{(1)}(i)}{a_{n-1}(i) - \boldsymbol{\alpha}_n^{(1)}(i)};\qquad b_n(i+1) = b_{n-1}(i)\,\frac{b_n(i) - \boldsymbol{\alpha}_n^{(1)}(i)}{b_{n-1}(i) - \boldsymbol{\alpha}_{n-1}^{(1)}(i)} \tag{71}$$

$$z_i\frac{b_n(i)}{\boldsymbol{\alpha}_n^{(1)}(i)} = z_i - a_n(i) + \boldsymbol{\alpha}_{n+1}^{(1)}(i) \tag{72}$$

where z_i are arbitrary constants.

One can also introduce the discrete analog of (6):

$$\Phi_n(i+1|z) = \big(1 - \boldsymbol{\alpha}_n^{(2)}(i)\big)\Phi_n(i|z) + z\boldsymbol{\alpha}_n^{(2)}(i)\Phi_{n-1}(i|z) \tag{73}$$

where $\boldsymbol{\alpha}_n^{(2)}(i)$ are some new unknown functions of the corresponding discrete indices. We refer to this evolution as to the second forward DBT. Substitution of (73) to (69) gives quite different system of the discrete evolution equations:

$$a_n(i+1) = a_n(i)\frac{1 - \boldsymbol{\alpha}_{n+1}^{(2)}(i)}{1 - \boldsymbol{\alpha}_n^{(2)}(i)});\qquad b_n(i+1) = b_{n-1}(i)\frac{\boldsymbol{\alpha}_n^{(2)}(i)}{\boldsymbol{\alpha}_{n-1}^{(2)}(i)} \tag{74}$$

$$a_n(i) + b_n(i)\,\frac{1 - \boldsymbol{\alpha}_n^{(2)}(i)}{\boldsymbol{\alpha}_n^{(2)}(i)} = z_i\,\frac{1}{1 - \boldsymbol{\alpha}_{n+1}^{(2)}(i)} \tag{75}$$

This system is the discrete counterpart of the continuum system (7).

Actually, in [3], four different discrete systems of the RTC equations were written. From our point of view, the additional evolutional systems result from the *backward Darboux-Bäcklund transformations* which are complimentary to those described above [6].

Continuum limit. Introducing some discrete shift of time $\Delta > 0$, one can rewrite all the equations describing the first forward DBT as follows:

$$a_n(t+\Delta) = a_{n-1}(t)\,\frac{a_n(t) - \boldsymbol{\alpha}_{n+1}^{(1)}(t)}{a_{n-1}(t) - \boldsymbol{\alpha}_n^{(1)}(t)} \tag{76}$$

$$b_n(t+\Delta) = b_{n-1}(t)\, \frac{b_n(t) - \boldsymbol{\alpha}_n^{(1)}(t)}{b_{n-1}(t) - \boldsymbol{\alpha}_{n-1}^{(1)}(t)} \tag{77}$$

After the rescaling $z_i \to g\,\Delta$, one gets from (72) $\boldsymbol{\alpha}_n^{(1)} \sim -g\Delta b_n/a_n$ and, in the continuum limit, $\Delta \to 0$ the last two equations lead directly to (4).

The analogous equations can be written for the second forward DBT but with z_i rescaled as $z_i \to \frac{1}{g\,\Delta}$. It is clear that, in the limit $\Delta \to 0$, one reproduces the continuum equations (7).

Limit to Toda chain. Now let us make the following expansion (compare with (5))

$$\begin{gathered} a_n(i) \simeq 1 - \epsilon p_n(i) \;\; ; \quad b_n(i) \simeq -\epsilon^2 R_n(i) \\ z \simeq 1 + \epsilon\lambda \;\; ; \quad z_i \simeq 1 + \epsilon\lambda_i \end{gathered} \tag{78}$$

Introduce also functions $\Psi_n(i)$

$$\Phi_n(i) \simeq \epsilon^n \Psi_n(i) \tag{79}$$

It is easy to see that (70) leads to the forward DBT for the non-relativistic Toda chain if one identifies

$$\boldsymbol{\alpha}_n^{(1)}(i) \simeq \epsilon A_n(i) \tag{80}$$

Indeed, in the limit $\epsilon \to 0$, one gets the standard Toda spectral and evolution equations.

There also exist some other interesting limits [6] leading to the modified discrete Toda equations [3].

Concluding remarks

From the point of view of studying the RTC hierarchy itself, the most promising representation is that describing the relativistic Toda hierarchy as a particular reduction of the two-dimensional Toda lattice hierarchy. However, even this quite large enveloping hierarchy is still insufficient. Loosly speaking, the Toda lattice is too "rigid" to reproduce both the continuous and discrete flows of the RTC. Therefore, one should embed the RTC into a more general system which admits more natural reductions.

This is done in the forthcoming publication [16], where we show that the RTC has a nice interpretation if considering it as a simple reduction of the two-component KP (Toda) hierarchy. It turns out that, in the framework of the 2-component hierarchy, the continuous AKNS system, Toda chain hierarchy *and* the discrete AKNS (which is equivalent to the RTC how we proved in this paper) can be treated on equal footing.

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